SEM 2, 2023-24: TOPOLOGY
END-SEMESTRAL EXAMINATION

Max score: 50 marks. Time: 3 hours.
Do any 5. Strike out the ones that you don't want graded. Otherwise only first 5 would be graded. State clearly any result that you use.

(1) For $k = \mathbb{R}$ or $\mathbb{C}$, let $\mathrm{M}_n(k)$ denote the set of all $n \times n$ matrices with values in k. Let $\mathrm{GL}_n(k) := \{A \in \mathrm{M}_n(k) : \det(A) \neq 0\}$. (8 + 2 marks)
  (a) Consider $\mathrm{M}_n(\mathbb{C})$ with euclidean topology under the identification $\mathrm{M}_n(\mathbb{C}) \simeq \mathbb{R}^{2n^2}$. Show that $\mathrm{GL}_n(\mathbb{C})$ is path-connected in the subspace topology.
  (b) Show that $GL_n(\mathbb{R})$ is not connected.

(2) Let $I_1 = I_2 = [-1, 1]$ be two copies of the closed interval. Let X be the quotient space of $I_1 \coprod I_2$ where each point of I_1 except 0 is identified with the corresponding point of I_2. Is X (4 × 2.5 marks)
  (a) compact?
  (b) Hausdorff?
  (c) connected?
  (d) metrizable?

Justify your answers.

(3) Recall that a topological space is said to be normal if its points are closed and any two disjoint closed sets can be separated by neighborhoods. (5+5 marks)
  (a) Prove that every compact, Hausdorff topological space is normal.
  (b) Prove that a connected normal space with more than one point must be uncountable.

(4) Let (X, τ) be a compact Hausdorff space. Let τ_1 be a topology on X which is strictly finer than τ and τ_2 a topology on X which is strictly coarser than τ. Show that (5+5 marks)
  (a) (X, τ_1) is not compact and
  (b) (X, τ_2) is not Hausdorff.

(5) Let X and Y be topological spaces and let $Y^X = \mathrm{Cont}(X, Y)$ denote the set of all continuous functions $X \to Y$. Consider Y^X with the compact-open topology, that is the topology generated by the subbasis consisting of sets of the form $S(C, U) = \{f : X \to Y | f(C) \subset U\}$, where C runs through all compact subsets of X and U runs through all open subsets of Y. (4+6 marks)
  (a) Let X be locally compact and Hausdorff. Show that the evaluation function $e : Y^X \times X \to Y$ defined by $e(f, x) = f(x)$ is continuous.
  (b) Let X, Z be Hausdorff and Z be locally compact. Show the exponential law: there is a natural function $Y^Z \times X \to (Y^Z)^X$, which is a homeomorphism.

Date: 22-04-2024.

(6) Prove that a connected open subset X of $\mathbb{R}^n$ is path-connected using the following steps: (4+4+2 marks)

(a) For any $x \in X$ let $U(x)$ be the set of all points in X that can be connected to x with a path. Prove that $U(x)$ is open, by showing for each $y \in U(x)$ there is an $\epsilon > 0$ such that $B_\epsilon(y) \subset U(x)$

(b) Prove that $U(x)$ is closed by showing its complement is open.

(c) Conclude X is path connected.